RESEARCH ARTICLE OPEN ACCESS

Common Fixed Point Theorems for a Pair of Self Maps on a Metric Space with a Partial Order Controlled By Rational Expressions

K.P.R.Sastry[1] , K.K.M.Sarma[2] , S.Lakshmana Rao[3*]
[1]*8-28-8/1, Tamil Street, Chinna Waltair Visakhapatnam-530 017, India.*
[2]*Professor, Department Of Mathematics, Andhra University, Visakhapatnam, 530 003, India.*
[3] *Research Scholar, Department Of Mathematics, Andhra University, Visakhapatnam-530 003, India.*

ABSTRACT
In this paper, We establish the existence of coincidence points for a pair of self maps on a partially ordered metric space, satisfying a contractive condition with rational expressions. We also investigate conditions for the existence and uniqueness of fixed points under certain conditions.
Keywords: α -admissible, (α, g) -admissible, triangular α -admissible, triangular (α, g) -admissible map f , (α, ψ) -contractive mapping. Mathematics Subject Classification (2010)}: 54H25, 47H10.

I. INTRODUCTION AND PRELIMINARIES

Banach contraction principle is one of the fundamental results on fixed point theory. Because of its importance in Non-linear analysis, a number of researches have improve and generalized this result. In 2012, B.Samet, C,Vetro and P.Vetro [1] introduced the concept of $(\alpha - \psi)$ -contractive maps, where α is an α -admissible mapping which is a new direction in the context of generalization of contraction maps and proved the existence of fixed points of such mappings in metric space setting. In 2013, E.Karapinar, P.Kumam and P.Salimi [4] introduced $(\alpha - \psi)$ -Mier-Keelar contractive mappings in the setting of complete metric space via triangular α -admissible mapping. In 2015, G.V.R.Babu, K.K.M.Sarma and V.A.Kumari [7] proved existence and uniqueness of common fixed points by for a pair of (f, g) of weak generalized (α, ψ) -contractive maps with rational expressions in partially ordered metric spaces.

Notation: Throughout this paper Ψ denotes the family of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that ψ is continuous on $[0, \infty)$ and $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each *t>0*, where ψ^n Is the n^{th} iterate of ψ .

Remark 1.1. Any function $\psi \in \Psi$ satisfies $\lim_{n \to \infty} \psi^n(t) = 0$ and $\psi(t) < t$, for any *t>0*.

Definition 1.2. (B.Samet, C.Vetro and P.Vetro [1]) Let (X, d) be a metric space $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ .We say that f is α -admissible,

if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$. (1.2.1)

In 2013, (E.Karapinar, P.Kumam and P.Salimi [4]) introduced the notion of a triangular α -admissible mapping as follows.

Definition 1.3. (E.Karapinar, P.Kumam and P.Salimi [4]) Let (X, d) be a metric space, $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that f is triangular α -admissible, if

(i) f is α -admissible ; and

(ii) $\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for any $x, y, z \in X$.

Definition 1.4. (B.Samet, C.Vetro and P.Vetro [1]) Let (X, d) be a metric space and $f : X \to X$ be a self map of X . Suppose there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(x, y) d(fx, fy) \leq \psi(d(x, y))$ for all $x, y \in X$. Then we say that f is a (α, ψ) -contractive mapping.

Remark 1.5. If $f : X \to X$ is a contraction with contractive constant *0<k<1*, then f is an (α,ψ) - contraction mapping, where $\alpha(x,y) \le 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \ge 0$.

Theorem 1.6. (B.Samet C.Vetro and P.Vetro [1]) Let (X,d) be a complete metric space and $f : X \to X$ be an (α,ψ) -contractive mapping. Suppose that

(i) f is $\alpha -$ admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \ge 1$; and

(iii) f is continuous.

Then f has a fixed point, that is, there exists $u \in X$ such that $fu = u$.

In 1977, D.S.Jaggi introduced 'rational type contraction mapping' as an extension of 'contraction maps' and proved the existence of fixed points of such mappings.

Theorem 1.7. (D.S.Jaggi [2]) Let f be a continuous self map defined on a complete metric space (X,d) . Suppose that f satisfies the following condition:

there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that

$$d(fx, fy) \le \alpha \frac{d(x, fx) d(y, fy)}{d(x,y)} + \beta d(x,y) \text{ for all } x, y \in X,\ x \ne y . \tag{1.7.1}$$

Then f has a fixed point in X .

Here we note that a mapping $f : X \to X$, X a metric space, that satisfies (1.7.1) is called a ` Jaggi contraction' map on X .

Later E.Karapinar and B.Samet [3] introduced generalized (α,ψ) -contractive mappings and proved fixed point results and their extensions to partially ordered metric spaces.

J.Harjani, B.Lopez and K.Sadarangani [10] extended theorem 1.7 to the context of partially ordered complete metric spaces.

Theorem 1.8. (J.Harjani, B.Lopez and K.Sadarangani [10]) Let $(X,\preceq)$ be a partially ordered set and (X,d) be a complete metric space. Let $f : X \to X$ be a non decreasing mapping such that

$$d(fx, fy) \le \alpha \frac{d(x, fx) d(y, fy)}{d(x,y)} + \beta d(x,y) \text{ for all } x, y \in X \tag{1.8.1}$$

with $x > y$ where $0 \le \alpha, \beta < 1$ with $\alpha + \beta < 1$.

Also assume either (i) f is continuous; (or)

(ii) if a non decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$ as $n \to \infty$ then $x_n \preceq x \ \forall n$.

Suppose there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. Then f has a fixed point.

A map f that satisfies the inequality (1.8.1) is called Jaggi contraction map in partially ordered metric spaces.

In 2013, M.Arshad, E.Karapinar and J.Ahmad [12] extended theorem 1.8 to almost Jaggi contraction type mapping in partially ordered metric spaces.

Note: Let $(X,\preceq)$ be a nonempty set and (X,d) be a metric space. Then $(X,d,\preceq)$ is called a partially ordered metric space.

Definition 1.9. (M.Arshad, E.Karapinar and J.Ahmad [12]) Let $(X,d,\preceq)$ be a partially ordered metric space. A self mapping f on X is called almost Jaggi contraction if it satisfies the following condition: there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and $L \ge 0$ such that

$$d(fx, fy) \le \alpha \frac{d(x, fx) d(y, fy)}{d(x,y)} + \beta d(x,y) + L.min\{d(x, fx), d(x, fy), d(y, fx)\} \tag{1.9.1}$$

for any distinct comparable $x, y \in X$.

Theorem 1.10. (M.Arshad, E.Karapinar and J.Ahmad [12]) Let $(X,d,\preceq)$ be a partially ordered complete metric space. Suppose that f is a continuous and non decreasing self map on X that satisfies the following inequality: there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and $L \ge 0$ such that

$$d(fx, fy) \le \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \beta d(x, y) + L.min\{d(x, fy), d(y, fx)\} \quad (1.10.1)$$

for any distinct comparable $x, y \in X$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq fx_0$.

Then f has a fixed point.

Remark 1.11. Since every almost Jaggi contraction satisfies the inequality (1.10.1), it follows that the conclusion of theorem 1.10 is valid under the replacement of condition (1.10.1) by almost Jaggi contraction in theorem 1.10.

In 1986, G.Jungck [8] introduced the concept of `compatible maps' as a generalization of `commuting maps' and proved the existence of fixed points in metric spaces. In 1998, G.Jungck and B.E.Rhoads [9] introduced the concept of `weakly compatible' maps as a generalization of `compatible maps'.

Definition 1.12. (G.Jungck [8]) Two self mappings f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Definition 1.13. (G.Jungck and B.E.Rhoades [9]) Two self mappings f and g of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points. i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Definition 1.14. (R.P.Pant [14]) Let (X, d) be a metric space and f, g be self maps of X . We say that f and g are 'reciprocally continuous' if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

The following well known lemma which we use in this paper can be easily established. However, a proof can be found in [5].

Lemma 1.15. (G.V.R.Babu and P.D.sailaja [5]) Suppose (X, d) be a metric space, $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$d(x_{m(k)}, x_{n(k)}) \ge \epsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$ and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$ (ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$

(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$ and (iv) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon$.

Definition 1.16. (L.Ciric, N.Cakic, M.R.Rajovic and J.S.Ume [11]) Suppose $(X, \preceq)$ is a partially ordered set and $f, g : X \to X$ are two self mappings of X . f is said by $g-$ non-decreasing if

For $x, y \in X$, $gx \preceq gy$ implies $fx \preceq fy$.

In 2014, G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [6] introduced the notion as (α, g) -admissibility of a self map on a metric space.

Definition 1.17. (G.V.R.Babu, K.K.M.Sarma and V.A.Kumari [6]) Let f, g be two self mappings on a metric space X . Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that the map f is (α, g) -admissible map if for $x, y \in X$, $\alpha(gx, gy) \ge 1 \Rightarrow \alpha(fx, fy) \ge 1$ (1.17.1)

In 2015, G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [7] introduced the concept namely ' f is a triangular (α, g) -admissible map'.

Definition 1.18. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [7]) Let (X, d) be a metric space and f, g be two self maps on X . Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that the map f is triangular (α, g) -admissible if

(i) f is (α, g) -admissible; and

(ii) $\alpha(gx, gy) \ge 1$, $\alpha(gy, gz) \ge 1 \Rightarrow \alpha(gx, gz) \ge 1$ for any $x, y, z \in X$. (1.18.1)

If $g = I_X$, the identity map of X in (1.18.1), then we call ` f is triangular $\alpha -$ admissible'.

Example1.19. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [7]). Let X = [0,3] with the usual metric.
Let A= $\Delta \cup \{(0,2),(3,0),(0,3),(2,3)\}$ and B $=\{(x,y) \in X \times X : x \neq y\} - \{(0,2),(3,0),(0,3),(2,3)\}$. where $\Delta = \{(x,x) / x \in X\}$.

We define $f, g : X \to X$ by $fx = \begin{cases} \frac{3x}{2} & if \;\; x \in [0,2] \\ 3 & if \;\; x \in (2,3] \end{cases}$ and $gx = \begin{cases} x & if \;\; x \in [0,2] \\ \frac{x+1}{2} & if \;\; x \in (2,3] \end{cases}$

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} 2e^{|x-y|} & if \;\; (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is a triangular (α, g) - admissible mapping. But g is not $(\alpha, f) -$ admissible mapping.

For, choose $x = 3$ and $y = 0$. In this case $fx = 3, fy = 0; gx = 2$ and $gy = 0$. Hence we have $\alpha(fx, fy) = \alpha(3,0) = 2e^3 \geq 1$.

But $\alpha(gx, gy) = \alpha(2,0) = 0 \not\geq 1$. Therefore g is not $(\alpha, f) -$ admissible mapping.

Example 1.20. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari)[7]. Let X = [0,2] with the usual metric.
Let A = $\Delta \cup \{(0,1),(1,2)\}$ and B = $\{(x,y) \in X \times X : x \neq y\} - \{(0,1),(1,2)\}$.

We define $f, g : X \to X$ We define $f, g : X \to X$ by

$fx = \begin{cases} x^2 & if \;\; x \in [0,1] \\ \frac{x}{2} & if \;\; x \in (1,2] \end{cases}$ and $gx = \begin{cases} x & if \;\; x \in [0,1] \\ 2 & if \;\; x \in (1,2] \end{cases}$

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} 2 & if \;\; (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is (α, g) - admissible mapping. But g is not (α, f) - admissible mapping.

For choosing $x = 0$ and $y = 2$. In this case $fx = 0, fy = 1, gx = 0$ and $gy = 2$.

Hence we have $\alpha(fx, fy) = \alpha(0,1) = 2 \geq 1$. But $\alpha(gx, gy) = \alpha(0,2) = 0 \not\geq 1$.

Therefore g is not (α, f) - admissible mapping.

Here we observe that f is not triangular $(\alpha, g) -$ admissible mapping.

For, by choosing $(x,y) = (0,1)$ and $(y,z) = (1,2)$

We have $\alpha(g0, g1) = \alpha(0,1) = 2 \geq 1$, $\alpha(g1, g2) = \alpha(1,2) = 2 \geq 1$ but $\alpha(g0, g2) = \alpha(0,2) = 0 \not\geq 1$.
Therefore condition (ii) of inequality (1.18.1) fails to hold. Therefore f is not triangular (α, g) - admissible mapping.

Example 1.21. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [7]). Let $X = \{1,2,3\}$ with the usual metric.
Let A $=\{(1,1),(2,2),(3,3),(1,2),(2,3),(3,2)\}$ and B $=\{(2,1),(3,1),(1,3)\}$.

We define $f, g : X \to X$ by $f1 = f3 = 3, f2 = 1; g1 = 1, g2 = 2$ and $g3 = 3$.

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} 2 & if \;\; (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is not (α, g) - admissible mapping and g is not (α, f) - admissible mapping.

For we choose $x = 1$ and $y = 2$, in this case $fx = 3, fy = 1, gx = 1$ and $gy = 2$.

Hence we have $\alpha(gx, gy) = \alpha(1,2) = 2 \geq 1$. But $\alpha(fx, fy) = \alpha(3,1) = 0 \not\geq 1$.

Therefore f is not (α, g) - admissible mapping.

Further, we choose $x = 1$ and $y = 3$, in this case $fx = fy = 3; gx = 1$ and $gy = 3$.

Hence we have $\alpha(fx, fy) = \alpha(3,3) = 2 \geq 1$. But $\alpha(gx, gy) = \alpha(1,3) = 0 \ngeq 1$.

Therefore g is not (α, f) - admissible mapping.

Example 1.22. Let $X = \{0,1,2,3\}$ with the usual metric.

Let A $=\{(0,0),(1,1),(2,2),(3,3),(3,1),(0,2),(0,3)\}$ and

B $=\{(0,1),(2,0),(1,0),(2,1),(1,2),(3,0),(1,3),(2,3),(3,2)\}$.

We define $f, g : X \to X$ by $f0 = f3 = 0,\ f1 = f2 = 2;\ g0 = g2 = 0, g1 = 1$ and $g3 = 3$.

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x, y) = \begin{cases} 2 & if \quad (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is not (α, g) - admissible mapping and hence f is not triangular (α, g) - admissible mapping.

For we choose $x = 2, y = 3$. We have $\alpha(gx, gy) = \alpha(g2, g3) = \alpha(0,3) = 2 \geq 1$.

But $\alpha(fx, fy) = \alpha(f2, f3) = \alpha(2,0) = 0 \ngeq 1$. Therefore f is not (α, g) - admissible mapping.

Hence f is not triangular (α, g) - admissible mapping.

In [7], it is mentioned that f is (α, g) - admissible mapping, which is not true.

Notation: Let f be a triangular (α, g) -admissible mapping and suppose $f(X) \subseteq g(X)$. Assume that there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $gx_{n+1} = fx_n$. Then $\alpha(gx_m, fx_n) \geq 1$ for all $m, n \in \mathbf{N}$ with $m < n$.

Definition 1.23. (G.V.R.Babu, K.K.M.Sarma and V.A.Kumari [6]) Let $(X, \preceq)$ be a partially ordered metric space and suppose that $f : X \to X$ is a mapping. If there exist two functions

$\alpha : X \times X \to [0,\infty)$, $\psi \in \Psi$ and $L \geq 0$ such that

$$\alpha(x, y)d(fx, fy) \leq \psi(M(x, y)) + L.N(x, y). \qquad (1.23.1)$$

where $M(x, y) = \begin{cases} max\{d(x,y), \dfrac{d(x,fx)d(y,fy)}{d(x,y)}, \dfrac{d(x,fy)d(y,fx)}{d(x,y)}, \\ \qquad \dfrac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{2d(x,y)}\} & if\ x\ and\ y\ are\ comparable. \\ 0 & if \quad x=y. \end{cases}$

and $N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx)\}$, $x, y \in X$ with $x \preceq y$. Then we say that f is weak generalized (α, ψ) -contractive map with rational expressions.

Note: Clearly, if f is Jaggi contraction. i.e., a map f that satisfies (1.8.1) with $\alpha + \beta < 1$ then it satisfies the inequality (1.23.1) with $\alpha(x, y) = 1 \ \forall \ x, y \in X$, $L = 0$ and $\psi(t) = (\alpha + \beta)t, t \geq 0$. Then f is weak generalized (α, ψ) -contractive map with rational expressions.

Example 1.24. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari[7]) Let $X = \{0,1,2\}$ with the usual metric. We define a partial order $\preceq$ on X by $\preceq := \{(0,0),(1,1),(2,2),(0,1),(0,2),(1,2)\}$

Let $A = \{(0,0),(0,2),(1,1),(2,2),(2,0),(1,2)\}$ and $B = \{(0,1),(1,0),(2,1)\}$.

We define $f : X \to X$ by $f0 = 2,\ f1 = 0$ and $f2 = 2$.

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x, y) = \begin{cases} \dfrac{3}{2} & if \quad (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is weak generalized (α, ψ) -contractive map with rational expressions.

Definition 1.25. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari[7]) Let $(X, \preceq)$ be a partially ordered metric space and let f and g be two self mappings on X. If there exist two functions

$\alpha : X \times X \to [0,\infty)$, $\psi \in \Psi$ and $L \geq 0$ such that

$$\alpha(gx, gy) d(fx, fy) \leq \psi(M(x,y)) + L.N(x,y). \quad (1.25.1)$$

Whenever *gx, gy* are comparable, where

$$M(x,y) = \begin{cases} max\{d(gx,gy), \dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)}, \dfrac{d(gx,fy)d(gy,fx)}{d(gx,gy)}, \\ \qquad \dfrac{d(gx,fx)d(gx,fy)+d(gy,fy)d(gy,fx)}{2d(gx,gy)}\}, \; if \; gx \neq gy. \\ 0 \qquad if \; gx = gy. \end{cases}$$

and $N(x,y) = \min\{d(gx,fx), d(gx,fy), d(gy,fx)\}$, $x, y \in X$ with $x \preceq y$.

Then we say that (f, g) is a pair of $weak\ generalized\ (\alpha, \psi)$ -contractive map with rational expressions.

If $g = I_X$, the identity map of X, in (1.25.1), then the inequality (1.25.1) reduces to (1.23.1) so that f is a $weak\ generalizid\ (\alpha, \psi)$ -contractive map with rational expressions.

Note: Clearly, a map f which satisfies (1.8.1) with $\alpha + \beta < 1$ also satisfies the inequality (1.25.1) with $\alpha(x,y) = 1 \, \forall x, y \in X, L = 0, g = I_X$ and $\psi(t) = (\alpha + \beta)t, \, t \geq 0$, Hence f is weak generalized (α, ψ) -contractive map with rational expressions. The following example satisfies inequality (1.25.1).

Example 1.26. Let $X = \{1, 2, 4\}$ with the usual metric. We define a partial order $\preceq$ on X by $\preceq := \{(1,1), (2,2), (4,4), (1,2), (1,4)\}$. Let $A = \{(1,1), (2,2), (4,4), (1,2), (1,4)\}$ and $B = \{(2,1), (4,1,), (2,4), (4,2)\}$. We define $f, g : X \to X$ by $f1 = f2 = 1, f4 = 2$ and $g1 = g2 = 4, g4 = 2$.

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} \frac{3}{2} & if \; (x,y) \in A \\ 0 & otherwise. \end{cases}$

and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{4}{5}t$.

Now we show that this example illustrates definition 1.25.

The following cases arise to verify the inequality (1.25.1).

$\underline{Case(i)}$: $x = 1$ and $y = 2$.

In this case $d(f1, f2) = 0$, $M(1,2) = 3$ and $N(1,2) = 0$.

$$\alpha(gx, gy) d(fx, fy) = \alpha(g1, g2) d(f1, f2) = 0 \leq \frac{12}{5} = \psi(3) + L.0$$

$$= \psi(M(1,2)) + L.N(1,2) = \psi(M(x,y)) + L.N(x,y) \text{ holds for any } L \geq 0$$

$\underline{Case(ii)}$: $x = 1$ and $y = 4$.

In this case $d(f1, f4) = 1$, $M(1,4) = 2$ and $N(1,4) = 1$.

$$\alpha(gx, gy) d(fx, fy) = \alpha(g1, g4) d(f1, f4) = 0 \leq \frac{8}{5} + L = \psi(2) + L.1$$

$$= \psi(M(1,4)) + L.N(1,4) = \psi(M(x,y)) + L.N(x,y) \text{ holds for any } L \geq 0$$

$\underline{Case(iii)}$: $x = 2$ and $y = 4$.

In this case $d(f2, f4) = 1$, $M(2,4) = 3$ and $N(2,4) = 1$.

$$\alpha(gx, gy)d(fx, fy) = \alpha(g2, g4)d(f2, f4) = 0 \le \frac{12}{5} + L = \psi(3) + L.1$$

$$= \psi(M(2,4)) + L.N(2,4) = \psi(M(x,y)) + L.N(x,y) \quad \text{holds for any } L \ge 0$$

Here we observe that the inequality (1.8.1) is also holds.

$\underline{Case(i)}$: $x = 1$ and $y = 2$.

In this case

$$d(fx, fy) = d(f1, f2) = 0 \le \beta = \alpha.0 + \beta = \alpha \frac{d(1, f1)d(2, f2)}{d(1,2)} + \beta d(1,2) = \alpha \frac{d(x, fx)d(y, fy)}{d(x,y)} + \beta d(x,y)$$

$\underline{Case(ii)}$: $x = 1$ and $y = 4$.

In this case

$$d(fx, fy) = d(f1, f4) = 1 \le 3\beta = \alpha.0 + 3.\beta = \alpha \frac{d(1, f1)d(4, f4)}{d(1,4)} + \beta d(1,4) = \alpha \frac{d(x, fx)d(y, fy)}{d(x,y)} + \beta d(x,y)$$

$\underline{Case(iii)}$: $x = 2$ and $y = 4$.

In this case

$$d(fx, fy) = d(f2, f4) = 1 \le \alpha + 2\beta = \alpha.1 + 2.\beta = \alpha \frac{d(2, f2)d(4, f4)}{d(2,4)} + \beta d(1,4) = \alpha \frac{d(x, fx)d(y, fy)}{d(x,y)} + \beta d(x,y)$$

$\therefore$ f is a Jaggi contraction with $\alpha = 0, \beta = \frac{1}{2}$ .Thus f is contraction with contraction constant $\frac{1}{2}$.

Note: G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [7] claim that the following example is an illustration of definition 1.25. However this example does not illustrate definition 1.25 in view of the following.

Example 1.27. (G.V.R Babu, K.K.M.Sarma, and V.A.Kumari [7]) Let $X = \{1, 2, 4, 6\}$ with the usual metric. We define a partial order $\preceq$ on X

by $\preceq := \{(1,1), (2,2), (4,4), (1,2), (6,6), (1,2), (1,4), (2,6), (1,6)\}$.

Let

$A = \{(1,1), (2,2), (4,4), (6,6), (1,2), (1,4), (2,6)\}$ and

$B = \{(2,1), (4,1,), (1,6), (6,1), (6,2), (2,4), (4,2), (4,6), (6,4)\}$.

We define $f, g : X \to X$ by $f1 = f2 = 1, f4 = f6 = 2$; $g1 = 1, g2 = g4 = 4, and\ g6 = 2$.

We define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x,y) = \begin{cases} \frac{3}{2} & if \quad (x,y) \in A \\ 0 & otherwise. \end{cases}$

and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{4}{5}t$.

By choosing $x = 1$, $y = 6$.

$$\alpha(gx, gy).d(fx, fy) = \alpha(g1, g6).d(f1, f6) = \alpha(1,2).d(1,2) = \frac{3}{2}, \quad \frac{4}{5} = \psi(1) + L.0 = \psi(M(x,y)) + L.N(x,y).$$

Hence the inequality (1.25.1) fails to hold.

II. MAIN RESULT

In the following, first we prove the existence of coincidence points of a pair (f, g) of weak generalized (α, ψ) - contractive maps with rational expressions.

Theorem 2.1. Let $(X, \preceq)$ be a poset and (X, d) be a metric space such that

$x \preceq y \preceq z \Rightarrow d(x,y) \le d(x,z)$, $d(y,z) \le d(x,z)$ $\forall$ $x,y,z \in X$. (2.1.1)

Let $f, g : X \to X$ be two self maps on X. Suppose $gx \preceq ggx$ $\forall$ $x \in X$, f is triangular (α, g) -admissible map and f is g -non-decreasing map. Suppose that there exist two functions $\alpha : X \times X \to [0,\infty)$, $\psi \in \Psi$ and $L \ge 0$ such that

$\alpha(gx, gy) \cdot d(fx, fy) \le \psi(M(x,y)) + L.N(x,y)$. (2.1.2)

whenever gx and gy are comparable and $gx \ne gy$, where

$M(x,y) =$

$$\max\{d(gx,gy), \frac{d(gx,fx)d(gy,fy)}{d(gx,gy)}, \frac{d(gx,fy)d(gy,fx)}{d(gx,gy)}, \frac{d(gx,fx)d(gx,fy)+d(gy,fy)d(gy,fx)}{2d(gx,gy)}\}$$

and $N(x,y)$ =min $\{d(gx,fx), d(gx,fy), d(gy,fx)\}$.

Also assume that

(i) $fX \subseteq gX$.

(ii) There exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \ge 1$ with $gx_0 \preceq fx_0$.

(iii) $g(X)$ is complete subset of X (that is $g(X)$ is a complete metric space).

(iv) If $\{gx_n\}$ is a non-decreasing sequence in gX such that $gx_n \to gx$ as $n \to \infty$ then $gx_n \preceq gx$ and $gx_n \preceq y$ for some $y \in gX$ and $\forall n \Rightarrow gx \preceq y$.

(v) Suppose $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_m) \ge 1$ whenever $n<m$, then $\alpha(x_n, x) \ge 1$.

Then f and g have a coincidence point.

Proof. Let $x_0 \in X$ be as in (ii), i.e., $\alpha(gx_0, fx_0) \ge 1$, with $gx_0 \preceq fx_0$.

since $fX \subseteq gX$, we choose $x_1 \in X$ such that $gx_1 = fx_0$ and in a similar way there exist x_2 such that $gx_2 = fx_1$.

By using a similar argument we choose a sequence $\{x_n\}$ in X with

$fx_n = gx_{n+1}$ for n=1,2,.... (2.1.3)

Since $gx_0 \preceq fx_0 = gx_1$ and f is g -nondecreasing

we have $fx_0 \preceq fx_1$ so that $gx_1 \preceq gx_2$.

Further since $gx_1 \preceq gx_2$ and f is g -nondecreasing

we have $fx_1 \preceq fx_2$, so that $gx_2 \preceq gx_3$.

Inductively, it follows that $gx_n \preceq gx_{n+1}$ for all n=1,2,3.... (2.1.4)

Now, $\alpha(gx_0, gx_1) = \alpha(gx_0, fx_0) \ge 1$.

Since f is (α, g) - admissible, we get $\alpha(fx_0, fx_1) \ge 1$, i.e., $\alpha(fx_0, fx_1) \ge 1$

By induction it can be shown that $\alpha(gx_n, gx_{n+1}) \ge 1$ for all n=1,2,3.... (2.1.5)

If $gx_{n+1} = gx_{n+2}$, for some n, then $gx_{n+1} = fx_{n+1}$ so that x_{n+1} is a coincidence point of f and g .

Now we assume that $gx_n \ne gx_{n+1}$, for n=1,2,3.... so that $d(gx_n, gx_{n+1}) > 0$ (2.1.6)

Now from (2.1.2), (2.1.4) and (2.1.5) we have

$$d(gx_{n+1}, gx_{n+2}) = 1.d(fx_n, fx_{n+1}) \le \alpha(gx_n, gx_{n+1}) d(fx_n, fx_{n+1}) \le \psi(M(x_n, x_{n+1})) + L.N(x_n, x_{n+1})$$ (2.1.7)

where

$$M(x_n, x_{n+1}) = max\{d(gx_n, gx_{n+1}), \frac{d(gx_n, fx_n)d(gx_{n+1}, fx_{n+1})}{d(gx_n, gx_{n+1})}, \frac{d(gx_n, fx_{n+1})d(gx_{n+1}, fx_n)}{d(gx_n, gx_{n+1})},$$
$$\frac{d(gx_n, fx_n)d(gx_n, fx_{n+1}) + d(gx_{n+1}, fx_{n+1})d(gx_{n+1}, fx_n)}{2d(gx_n, gx_{n+1})}\}$$

$$= max\{d(gx_n, gx_{n+1}), \frac{d(gx_n, gx_{n+1})d(gx_{n+1}, gx_{n+2})}{d(gx_n, gx_{n+1})}, \frac{d(gx_n, gx_{n+1})d(gx_{n+1}, gx_{n+1})}{d(gx_n, gx_{n+1})},$$
$$\frac{d(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2})d(gx_{n+1}, gx_{n+1})}{2d(gx_n, gx_{n+1})}\}$$

$$= max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), 0, \frac{d(gx_n, gx_{n+2})}{2}\}$$

$M(x_n, x_{n+1}) = max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\}$ and

$N(x_n, x_{n+1}) = min\{d(gx_n, fx_n), d(gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)\}$

$= min\{d(gx_n, gx_{n+1}), d(gx_n, gx_{n+2}), d(gx_{n+1}, gx_{n+1})\} = 0$

From (2.1.7) we have

$$d(gx_{n+1}, gx_{n+2}) \leq \psi(max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\} + L.0 \quad (2.1.8)$$

Suppose $d(gx_n, gx_{n+1}) < d(gx_{n+1}, gx_{n+2})$

then from (2.1.8)

$d(gx_{n+1}, gx_{n+2}) \leq \psi(d(gx_{n+1}, gx_{n+2})) < d(gx_{n+1}, gx_{n+2})$ a contradiction.

Hence $d(gx_n, gx_{n+1}) \geq d(gx_{n+1}, gx_{n+2})$, so that (2.1.8) gives

$$d(gx_{n+1}, gx_{n+2}) \leq \psi(d(gx_n, gx_{n+1})) \text{ for all } n \quad (2.1.9)$$
$$< d(gx_n, gx_{n+1}) \ \forall \text{ n}.$$

Thus it follows that $\{d(gx_n, gx_{n+1})\}$ is strictly decreasing sequence of positive numbers that converges to a limit $r \geq 0$

i.e. $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = r \geq 0$

Now we show that r=0

From(2.1.9) $d(gx_{n+1}, gx_{n+2}) \leq \psi(d(gx_n, gx_{n+1}))$

$\leq \psi(\psi(d(gx_{n-1}, gx_n))) = \psi^2(d(gx_{n-1}, gx_n))$

$\leq$ ....

$\leq$ ....

$\psi^n(d(gx_0, gx_1)) \to 0$ as $n \to \infty$, $since \psi \in \Psi$. (2.1.10)

Hence $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0$.

Now we show that $\{gx_n\}$ is a Cauchy sequence in X .

For positive integers n and k we have

$$d(gx_n, gx_{n+k}) \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + ... + d(gx_{n+k-1}, gx_{n+k})$$
$$\leq \psi(d(gx_{n-1}, gx_n)) + \psi(d(gx_n, gx_{n+1})) + ... + \psi(d(gx_{n+k-2}, gx_{n+k-1}))$$
$$\leq \psi^n(d(gx_0, gx_1)) + \psi^{n+1}(d(gx_0, gx_1)) + ... + \psi^{n+k-1}(d(gx_0, gx_1)) \to 0$$

(by (2.1.10) and $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each *t >0).*

Therefore $\{gx_n\}$ is a Cauchy sequence.

Since $g(X)$ is complete, there exists $z \in g(X)$, such that $gx_n \to z = gx$, for some $x \in X$.

Hence $fx_n = gx_{n+1} \to gx$ (2.1.11)

Now we show that $fx = gx$.

Suppose $gx \neq fx$.

From(iv), since $\{gx_n\}$ is a non-decreasing sequence and $gx_n \to gx$ (2.1.12)

we have $gx_n \preceq gx \preceq fx \quad \forall\ n$

Now $gx \preceq ggx \Rightarrow fx \preceq fgx$. $(\because f$ is g non-decreasing)

We observe that $gx_n \neq gx$, for every n

Now $\alpha\ (gx, gx_n)\ d(fx, fx_n) \leq \psi\ (M(x, x_n)) + L.N(x, x_n)$ (2.1.13)

Now $M(x, x_n) =$

$$max\{d(gx, gx_n), \frac{d(gx, fx)d(gx_n, fx_n)}{d(gx, gx_n)}, \frac{d(gx, fx_n)d(gx_n, fx)}{d(gx, gx_n)}, \frac{d(gx, fx)d(gx, fx_n) + d(gx_n, fx_n)d(gx_n, fx)}{2d(gx, gx_n)}\}$$

Now $\frac{d(gx, fx)d(gx_n, fx_n)}{d(gx, gx_n)} \leq d(gx, fx) \leq d(gx_n, fx)$ (2.1.14)

$\frac{d(gx, fx_n)d(gx_n, fx_n)}{d(gx, gx_n)} \leq 1.\ d(gx_n, fx)$ (2.1.15)

Now $\frac{d(gx, fx)d(gx, fx_n) + d(gx_n, fx_n)d(gx_n, fx)}{2d(gx, gx_n)} \leq$

$$d(gx, fx)\ \frac{d(gx, fx_n)}{2d(gx, gx_n)} + \frac{d(gx_n, fx_n)}{d(gx, gx_n)} . \frac{d(gx_n, fx)}{2}$$

$$\leq d(gx, fx). \frac{d(gx, fx_n)}{2d(gx, gx_n)} + \frac{d(gx_n, fx)}{2}$$

$$\leq \frac{d(gx, fx)}{2} + \frac{d(gx_n, fx)}{2} = d(gx, fx)$$

$\leq d(gx_n, fx)$ (2.1.16)

Therefore $M(x, x_n) \leq max\{d(gx, gx_n), d(gx_n, fx), d(gx_n, fx), d(gx_n, fx)\}$

$\therefore\ M(x, x_n) \leq d(gx_n, fx)$ (2.1.17)

and $N(x, x_n) = min\{d(gx, fx), d(gx, fx_n), d(gx_n, fx)\}$

$= d(gx, fx_n)$

$= d(gx, gx_{n+1})$ for large n. (2.1.18)

From (2.1.13)

$d(fx, fx_n) \leq \alpha(gx, gx_n).d(fx, fx_n) \leq \psi\ (d(gx_n, fx)) + L.d(gx, fx_n)$.

Therefore $d(fx, fx_n) \leq \psi\ (d(gx, fx)) + 0 < d(gx, fx)$

$\therefore\ d(fx, gx_{n+1}) = d(fx, gx) < d(gx, fx)$

a contradiction.

Hence $gx = fx$.

Therefore x is a coincidence point of f and g.

Corollary 2.2. Let $(X, \preceq)$ be a poset and (X, d) be a complete metric space such that

$x \leq y \leq z \Rightarrow d(x, z) \leq d(x, z)$, and $d(y, z) \leq d(x, z)\ \forall\ x, y, z \in X$ (2.2.1).

Let $f : X \to X$ be a non-decreasing map. Suppose $\alpha : X \times X \to [0,\infty)$ is such that $\alpha(x,y) \geq 1 \Rightarrow \alpha(fx, fy)$ and $\alpha(x,y) \geq 1$, $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1 \ \forall x,y,z \in X$. (2.2.2)

Suppose $M(x,y) =$

$$\max \{d(x,y), \frac{d(x,fx)d(y,fy)}{d(x,y)}, \frac{d(x,fy)d(y,fx)}{d(x,y)}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{2d(x,y)}\}$$

(2.2.3) $\forall x,y \in X$ with $x \neq y$

and $N(x,y) = \min\{d(x,y), d(x,fy), d(y,fx)\}$ (2.2.4)

Suppose there exist $\psi \in \Psi$ such that $\alpha(x,y).d(fx,fy) \leq \psi(M(x,y)) + L.N(x,y)$

Whenever x and y are comparable and $x \neq y$, where $L \geq 0$

Assume that,

there exists $x_0 \in X$. such that $\alpha(x_0, fx_0) \geq 1$, with $x_0 \preceq fx_0$ (2.2.6)

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$ then $x_n \preceq x$

and $x_n \preceq y$ for some $y \in X$ and $\forall$ $n \Rightarrow x \preceq y$ (2.2.7)

and $\alpha(x_n, x_m) \geq 1$ whenever *n<m*, then $\alpha(x_n, x) \geq 1$ (2.2.8)

Then f has a fixed point.

Proof. Take $g = I_X$ in theorem 2.1.

Corollary 2.3. Let $(X, \preceq)$ be a poset and (X, d) be a complete metric space such that (2.2.1) holds. Let $f : X \to X$ be a non-decreasing map. Suppose $\alpha : X \times X \to [0,\infty)$ satisfies (2.2.2). Suppose there exist constants $k \in (0,1)$ and $L \geq 0$ such that $\alpha(x,y).d(fx,fy) \leq k(M(x,y)) + L.N(x,y)$ whenever x and y are comparable and $x \neq y$. Further assume that (2.2.6), (2.2.7) and (2.2.8) hold.

Then f has a fixed point.

Proof. Take $\psi(t) = kt$ in corollary 2.2.

Theorem 2.4. In addition to the hypotheses of theorem 2.1, suppose z is another coincidence point of f and g . Suppose *gx* and *gz* are comparable. Then $\alpha(gx, gz) \geq 1 \Rightarrow gx = gz$ and $\alpha(gz, gx) \geq 1 \Rightarrow gx = gz$.

Proof. Suppose $\alpha(gx, gz) \geq 1$. Further suppose that $gz \neq gx$ so that $d(gz, gx) > 0$

Now $d(gz,gx) \leq \alpha(gz,gx).d(gz,gx) \leq \psi(M(z,x)) + L.N(z,x)$ (2.4.1)

where $M(z,x) =$

$$\max \{d(gz,gx), \frac{d(gz,fz)d(gx,fx)}{d(gz,gx)}, \frac{d(gz,fx)d(gx,fz)}{d(gz,gx)}, \frac{d(gz,fz)d(gz,fx)+d(gx,fx)d(gx,fz)}{2d(gz,gx)}\}$$

$$= \max\{d(gz,gx), 0, d(gx,gz), \frac{d(gx,gx)}{2}\} = d(gz,gx)$$

$N(z,x) = \min\{d(gz,fz), d(gz,fx), d(gx,fz)\} = 0$

From (2.4.1), $d(gz,gx) \leq \psi(d(gz,gx)) + 0 < d(gz,gx)$ a contradiction.

$\therefore \ gz = gx$.

In the following theorem we prove the existence and uniqueness of common fixed point of f and g .

Theorem 2.5. In addition to the hypothesis of theorem 2.1, suppose that f and g are weakly compatible. Then f and g have a common fixed point, say z . If u is any common fixed point of f and g comparable with z , then either $\alpha(u,z) < 1$ or $u = z$

Proof. By theorem 2.1 f and g have coincidence point.

Suppose x is a coincidence point of f and g so that $fx = gx$

Since f and g are weakly compatible, we have $fgx = gfx$

so that, $fgx = g(gx)$ which implies that $z = gx$ is a coincide point of f and g.

Now $gx \preceq ggx = gz$.

Suppose $\alpha(gx, gz) \geq 1$. Then $z = gx = gz$ (by theorem 2.4)

$\Rightarrow$ $z = gx$ is a fixed point of g

Now $fgx = gfx = g(gx) = gz = gx$

Therefore $fz = z$

$\therefore$ z is a fixed point of f.

$\therefore$ z is a common fixed point of f and g.

Suppose u is also common fixed point of f and g and u is comparable with z.

If $\alpha(u, z) < 1$, we are through.

Now suppose $\alpha(u, z) \geq 1$. Then $\alpha(gu, gz) = \alpha(u, z) \geq 1 \Rightarrow gu = gz$ (by theorem 2.4)

Therefore $u=gu=gz=z$

Therefore $u=z$.

REFERENCES

[1]. B.Samet, C.Vetro and P.Vetro, Fixed point theorem for (α, ψ)-contractive type mappings, Nonlinear Anal. 75(2012), 2154-2165.

[2]. D.S.Jaggi, Some unique fixed point's theorems.Indian J.Pure and Appl.Math.and Math.Sci.9, 1986, 771-779.

[3]. E.Karapinar and B.Samet, Generalized (α, ψ)-contractive type mappings and related fixed points theorems with applications.Abst.and Appl.Anal.2012, ID 793486, 17pages.

[4]. E.Karapinar, P.Kumam and P.Salimi, On (α, ψ)-Meir-Keeler-contractive mappings, Fixed Point Theory Appl.,2013, Article ID 34(2013).

[5]. G.V.R.Babu, P.D.Sailaja, A fixed point theorem of generalized weak contractive maps in orbitally complete metric spaces, Thai Journal of Mathematics. 9(1) (2011),1-10.

[6]. G.V.R.Babu, K.K.M.Sarma and V.A.Kumari,Common fixed points of (α, η)-Geraghty contraction maps, J.of Adv.Res.in Pure Math. 7(2), 2015, pp.43-53.

[7]. G.V.R.Babu, K.K.M.Sarma and V.A.Kumari,Common fixed points of weak generalized (α, ψ)-contractive maps in partially ordered metric spaces, Asian Journal of math.and App.vol 2015, article ID ama0225,20pages.

[8]. G.Jungck, Compatible mappings and common fixed points, Int.J.Math.and Math.Sci.9, 1986, 771-779.

[9]. G.Jungck and B.E.Rhoades, Fixed Point for set valued function without continuity, Indian J.Pure Appl.Math.,29(3)(1998)227-238.

[10]. J.Harjani, B.Lopez and K.Sadarangani, A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Abstract and Appl.Anal.,2010,1-8(2010).

[11]. L.Ciric, N.Cakic, M.Rajovic and J.S.Ume, Monotone generalized nonlinear contractions in a partially ordered metric spaces, Fixed Point Theorey Appl., 2008 Article ID 131294.

[12]. M.Arshad, E.Karapinar and J.Ahmad, Some unique Fixed Point Theorems for rational contractions in a partially ordered metric spaces, Journal of Inequalities and Appl.,Article ID 307234,(2013).

[13]. N.Hussain, P.Salimi and A.Latif, Modified (α, ψ)-contractive type mappings with applications,Fixed Point Theorey Appl.,2013, Article ID 151, doi:10.1186/1687-1812-2013-212.

[14]. R.P.Pant, A Common fixed point theorem under a new condition. Indian J.Pure Appl.Math. 30(2),(1999),147-152.